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# Many-time photocount distributions

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**Abstract.** The generating function is obtained for $N$ photocounts for a gaussian optical field with a lorentzian profile without any restriction to the intervals during which the photodetectors are open. The method may be generalized to arbitrary spectral profiles using the method of Srinivasan and Sukavanam.

## 1. Introduction

The statistical specification of an optical field requires the complete density matrix in the Fock space for the quantum optical case or the field ensemble density for the classical optical case. But the directly measured quantities in the optical case are neither the photon occupation numbers nor the intensities but photocounts. It is therefore of great interest to make a systematic analysis of photocount distributions.

The analysis of photocount correlations of 'fluctuating' beams of light (ie statistical optical fields) has been presented recently by Dialetis (1969) as well as Jakeman (1970). Dialetis (1969) presents a method to obtain the explicit generating function of the $N$ time photon counts. He assumes that all the counting is done in an interval $T$. The same holds in our discussion here but the present method can be extended to cover cases where the counting is not thus confined in time, whether the individual sample times are equal or unequal. Jakeman (1970) has analysed the problem for a mixture consisting of an incoherent gaussian component and a single frequency coherent beam and has obtained an explicit expression for the generating function of the double distribution corresponding to two disjoint and equal time intervals. In this paper we develop a method leading to the explicit determination of $N$ time joint distribution of an incoherent beam corresponding to general intervals which may or may not overlap with one another. Cantrell (1971) has derived the $N$ time generating function but his results correspond to cross-spectrally pure light while the counting times of the different detectors are required to be equal. Like Jakeman (1970) we discuss the counts on a single detector counting during $N$ intervals. As such the discussion presented here does not demand cross-spectral purity and holds even for unequal counting times provided all of them fall within an interval $T$. Even when they do not, the treatment of the problem is a simple extension of the following discussion.

## 2. Photocount correlations for gaussian beams

The generating function corresponding to the $N$ fold probability density function of a gaussian field is given by

$$Q(S_1, S_2, \dots S_N) = E\left[\exp\left(\sum_{i=1}^{N} -S_i W_i\right)\right] \tag{2.1}$$

where

$$W_i = \int_{t_i}^{t_i+T_i} V^*(t)V(t)\,\mathrm{d}t \tag{2.2}$$

where $V(t)$ is the random complex field corresponding to the fluctuating analytic signal and the intervals $(t_i, t_i + T_i)$ $(i = 1, 2, \dots N)$ are non-overlapping. The non-overlapping nature does not restrict the general problem of the determination of photocounts in overlapping intervals since the generating function can always be cast in the form (2.1) with a new choice of variables $S_1, S_2, \dots S_N$.

We next expand $V(t)$ into an orthonormal set of functions over the $L_2$ space corresponding to the intervals $(t_i, t_i + T_i)$.

$$V(t) = \sum a_n \phi_n(t) \tag{2.3}$$

where the normalization condition is

$$\sum S_i \int_{t_i}^{t_i+T_i} \phi_m(t)\phi_n^*(t)\,\mathrm{d}t = \delta_{mn}. \tag{2.4}$$

The quantities $\{a_i\}$ are statistically independent and satisfy the relations

$$\langle a_i a_j^* \rangle = \langle m_i \rangle \delta_{ij} \tag{2.5}$$

$$p(a_k) = \frac{\exp(-|a_k|^2/\langle m_k \rangle)}{\pi \langle m_k \rangle} \tag{2.6}$$

so that we have

$$\sum S_i W_i = \sum_m |a_m|^2. \tag{2.7}$$

The autocorrelation of the complex $V$ field is given by

$$\langle V(t)V^*(t')\rangle = \sum_{k,j} \langle a_k a_j^* \rangle \phi_k(t)\phi_j^*(t'). \tag{2.8}$$

From equations (2.4) and (2.8) we find that

$$\left(\sum S_i \int_{t_i}^{t_i+T_i}\right) \langle V(t)V^*(t')\rangle \phi_l(t')\,\mathrm{d}t' = \langle m_l \rangle \phi_l(t). \tag{2.9}$$

Using the normalization of Jakeman and Pike (1968)

$$\begin{aligned} \langle V^*(t)V(t)\rangle &= \frac{\langle E \rangle}{T} \\ \lambda_k &= \frac{T}{\langle E \rangle}\langle m_k \rangle \end{aligned} \tag{2.10}$$

and the stationary nature of the $V$ field

$$\langle V(t)V^*(t')\rangle = \frac{\langle E\rangle}{T} g^{(1)}(t-t')$$

we obtain

$$\left(\sum S_i \int_{t_i}^{t_i+T_i}\right) g^{(1)}(t-t')\phi_l(t')\,\mathrm{d}t' = \lambda_l(S_1, S_2, \ldots S_N)\phi_l(t). \tag{2.11}$$

The determination of the eigenvalues $\lambda_l$ is the main objective of the analysis. We can then obtain the generating function $Q(S_1, S_2, \ldots S_N)$ which is now given by

$$Q(S_1, S_2, \ldots S_N) = \left\langle \exp\left(-\sum |a_k|^2\right)\right\rangle = \prod_k \left(1 + \frac{\langle E\rangle\lambda_k}{T}\right)^{-1}. \tag{2.12}$$

However, we show that it is possible to obtain $Q(S_1, S_2, \ldots S_N)$ directly by evaluating the infinite product given by the right-hand side of (2.12).

## 3. Laplace transform solution of the eigenfunctions

For notational convenience, we drop the suffix $l$ from now on. Thus equation (2.11) can be written as

$$\left(\sum S_i \int_{t_i}^{t_i+T_i}\right) \exp(-\Gamma|t-t'|)\phi(t')\,\mathrm{d}t' = \lambda\phi(t) \tag{3.1}$$

where we have allowed the autocorrelation function to correspond to a lorentzian profile of halfwidth $\Gamma$. Defining

$$\psi_i(p) = \int_{t_i}^{t_i+T_i} \phi(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t$$

and allowing $t_1 < t_2 \ldots < t_N$, we obtain after some calculation

$$\begin{aligned}\psi_m(p) = {} & [\lambda(\Gamma^2-p^2)-2\Gamma S_m]^{-1}\Big((\Gamma-p)\exp[-(\Gamma+p)t_m]\{1-\exp[-(\Gamma+p)T_m]\} \\ & \times \sum_{i=1}^{m-1} S_i\psi_i(-\Gamma) - S_m(\Gamma+p)\exp[(\Gamma-p)t_m]\psi_m(\Gamma) \\ & - S_m(\Gamma-p)\exp[-(\Gamma+p)(t_m+T_m)]\psi_m(-\Gamma) \\ & + \exp[(\Gamma-p)t_m](\Gamma+p)\{-1+\exp[(\Gamma-p)T_m]\} \sum_{i=m+1}^{N} S_i\psi_i(\Gamma)\Big).\end{aligned} \tag{3.2}$$

Next we observe that $\psi_m(p)$ is an entire function in the complex $p$ plane. However, the denominator $[\lambda(\Gamma^2-p^2)-2\Gamma S_m]$ occurring in (3.2) has two zeros at $p = \pm p_m$ where

$$p_m = +\left(\Gamma^2 - \frac{2\Gamma}{\lambda} S_m\right)^{1/2}. \tag{3.3}$$

In order that $\psi_m(p)$ shall not possess any singularity in the finite part of the $p$ plane, we must have

$$0 = (\Gamma - p_m)\exp[-(\Gamma + p_m)t_m]\{1 - \exp[-(\Gamma + p_m)T_m]\}\sum_{i=1}^{m-1} S_i\psi_i(-\Gamma)$$

$$- S_m(\Gamma + p_m)\exp[(\Gamma - p_m)t_m]\psi_m(\Gamma)$$

$$- S_m(\Gamma - p_m)\exp[-(\Gamma + p_m)(t_m + T_m)]\psi_m(-\Gamma)$$

$$+ \exp[(\Gamma - p_m)t_m](\Gamma + p_m)\{-1 + \exp[(\Gamma - p_m)T_m]\}\sum_{i=m+1}^{N} S_i\psi_i(\Gamma)$$

$$m = 1, 2, \ldots N. \tag{3.4}$$

We have another set of equations obtained by replacing $p_m$ by $-p_m$. Thus we have a set of $2N$ equations homogeneous and linear in $2N$ unknowns $\{\psi_i(\pm\Gamma)\}$. The set of equations can be written in the form

$$A\Psi = 0 \tag{3.5}$$

where $A$ is the $2N \times 2N$ matrix whose elements are given by

$$A_{ij}\begin{cases} = b_iS_j & j < i \\ = c_iS_i & j = i \\ = 0 & i < j < N+i \\ = d_iS_i & j = N+i \\ = e_iS_{j-N} & j > N+i \end{cases} \tag{3.6}$$

$$\left.\begin{aligned} b_i &= (\Gamma - p_i)\exp[-(\Gamma + p_i)t_i]\{1 - \exp[-(\Gamma + p_i)T_i]\} \\ c_i &= (\Gamma + p_i)\exp[(\Gamma - p_i)t_i] \\ d_i &= -(\Gamma - p_i)\exp[-(\Gamma + p_i)(t_i + T_i)] \\ e_i &= \exp[(\Gamma - p_i)t_i](\Gamma + p_i)\{\exp[(\Gamma - p_i)T_i] - 1\} \qquad (1 \leqslant i \leqslant N) \end{aligned}\right. \tag{3.7}$$

$$\left.\begin{aligned} b_{N+i} &= (\Gamma + p_i)\exp[-(\Gamma - p_i)t_i]\{1 - \exp[-(\Gamma - p_i)T_i]\} \\ c_{N+i} &= -(\Gamma - p_i)\exp[(\Gamma + p_i)t_i] \\ d_{N+i} &= -(\Gamma + p_i)\exp[-(\Gamma - p_i)(t_i + T_i)] \\ e_{N+i} &= \exp[(\Gamma + p_i)t_i](\Gamma - p_i)\{-1 + \exp[(\Gamma + p_i)T_i]\} \qquad (1 \leqslant i \leqslant N) \end{aligned}\right. \tag{3.8}$$

and $\Psi$ is a $2N$ dimensional vector with components

$$\psi_1(-\Gamma), \psi_2(-\Gamma)\ldots\psi_N(-\Gamma), \psi_1(\Gamma), \psi_2(\Gamma)\ldots\psi_N(\Gamma).$$

In order that (3.5) has a nontrivial solution, we must have

$$\det A = 0 \tag{3.9}$$

an equation which determines the eigenvalues $\lambda$ of the Fredholm integral equation (2.11). If we denote $\det A$ by $F(1/\lambda)$, it is easy to see that $F(1/\lambda)$ is not an entire function of its argument since it does not return to its original value if we go round any arbitrary closed contour containing the origin of the $\xi\,(\equiv 1/\lambda)$ plane. However it is easy to see that $F(\xi)$ is an analytic function of its argument in any bounded domain of the cut $\xi$

plane. Thus it is easy to render $F(\xi)$ entire by multiplying by an appropriate factor (see for example Srinivasan and Sukavanam 1972). For instance

$$P(\xi) = F(\xi)/p_1 p_2 \dots p_N \tag{3.10}$$

is an entire function of $\xi$ and its zeros are the eigenvalues of the basic integral equation (2.11).

We can obtain a representation of $P(\xi)$ by using the Hadamard–Weierstrass theorem relating to the canonical representation of an entire function. Since the order of $P(\xi)$ is half, we easily obtain

$$P(\xi) = P(0) \prod_k (1 - \xi/\xi_k) \tag{3.11}$$

where the constant $P(0)$ is determined using the explicit relation (3.10). Comparing (3.11) with (2.12), we obtain

$$Q(S_1, S_2, \dots S_N) = \frac{P(0)}{P(-\langle E \rangle/T)}. \tag{3.12}$$

This completes the determination of $Q(S_1, S_2, \dots S_N)$ for non-overlapping counting intervals. As for overlapping counting intervals, the explicit results can be obtained by making appropriate modifications as mentioned in the introduction. As long as the light is cross-spectrally pure the foregoing discussion covers even the case where $N$ detectors are present. In this case, the cross-spectral purity implies that the field $V(t)$ 'seen' by all the detectors is the same except for translation of the time variable. However, if there are $N$ detectors and the light is not cross-spectrally pure, each of the detectors will 'see' a field $V_i(t)$ $(i = 1, \dots N)$. Each $V_i(t)$ can be expanded in the interval $(t_i, t_i + T_i)$ in a manner similar to equation (2.3) and $g^{(1)}(t-t')$ in equation (2.11) will be replaced in each interval $(t_i, t_i + T_i)$ by $g_i^{(1)}(t-t')$, the normalized autocorrelation for the $i$th detector. With this modification, the whole procedure can be repeated and the generating function obtained for this case.

## 4. An example: the two-time distribution for gaussian–lorentzian light

As an illustration of the method outlined above, we now discuss the case of two-time counting with a single gaussian–lorentzian beam of mean intensity $\bar{I} = \langle E \rangle/T$. We take the two sampling times $T_1$ and $T_2$ to be unequal but assume that both the samples occur within the interval $(0, T)$. Also the sampling times are taken to be non-overlapping. The integral equation (3.1) now takes the form

$$s_1 \int_{t_1}^{t_1+T_1} \exp(-\Gamma|t-t'|)\Phi(t)\,\mathrm{d}t + s_2 \int_{t_2}^{t_2+T_2} \exp(-\Gamma|t-t'|)\Phi(t)\,\mathrm{d}t = \lambda\Phi(t') \tag{4.1}$$

where $\Phi$ is related to $\phi$ of equation (3.1) through the relation

$$\Phi(t) = \phi(t)\exp(\mathrm{i}\omega_0 t), \tag{4.2}$$

$\omega_0$ being the centre frequency of the beam. Now $t'$ can be either in the interval $(t_1, t_1 + T_1)$ or in the interval $(t_2, t_2 + T_2)$. Accordingly, we define two Laplace transforms

$$\hat{\psi}_1(p) = \int_{t_1}^{t_1+T_1} \Phi(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t \tag{4.3}$$

and

$$\hat{\psi}_2(p) = \int_{t_2}^{t_2+T_2} \Phi(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t. \tag{4.4}$$

We take $t_1 < t_2$. First we proceed to derive an expression for $\hat{\psi}_1(p)$. Taking the Laplace transform (as defined by (4.3)) of both sides of equation (4.1) we have

$$s_1 \int_{t_1}^{t_1+T_1} \mathrm{e}^{-pt'}\,\mathrm{d}t' \int_{t_1}^{t_1+T_1} \exp(-\Gamma|t-t'|)\Phi(t)\,\mathrm{d}t + s_2 \int_{t_1}^{t_1+T_1} \mathrm{e}^{-pt'}\,\mathrm{d}t' \times \int_{t_2}^{t_2+T_2} \exp(-\Gamma|t-t'|)\Phi(t)\,\mathrm{d}t = \lambda\hat{\psi}_1(p)$$

that is,

$$s_1 \int_{t_1}^{t_1+T_1} \mathrm{e}^{-pt'}\,\mathrm{d}t' \left(\int_{t_1}^{t'} \exp[-\Gamma(t'-t)]\Phi(t)\,\mathrm{d}t + \int_{t'}^{t_1+T_1} \exp[-\Gamma(t-t')]\Phi(t)\,\mathrm{d}t\right) + s_2 \int_{t_1}^{t_1+T_1} \mathrm{e}^{-pt'}\,\mathrm{d}t' \int_{t_2}^{t_2+T_2} \exp[-\Gamma(t-t')]\Phi(t)\,\mathrm{d}t = \lambda\hat{\psi}_1(p)$$

which leads to the equation

$$\begin{aligned}\hat{\psi}_1(p) = {} & [\lambda(\Gamma^2-p^2)-2\Gamma s_1]^{-1}(-s_1(\Gamma+p)\exp[(\Gamma-p)t_1]\hat{\psi}_1(\Gamma) \\ & -s_1(\Gamma-p)\exp[-(\Gamma+p)(t_1+T_1)]\hat{\psi}_1(-\Gamma) \\ & +\exp[(\Gamma-p)t_1](\Gamma+p)\{-1+\exp[(\Gamma-p)T_1]\}s_2\hat{\psi}_2(\Gamma)).\end{aligned} \tag{4.5}$$

Similarly,

$$\begin{aligned}\hat{\psi}_2(p) = {} & [\lambda(\Gamma^2-p^2)-2\Gamma s_2]^{-1}(s_1(\Gamma-p)\exp[-(\Gamma+p)t_2]\{1-\exp[-(\Gamma+p)T_2]\} \\ & \times\hat{\psi}_1(-\Gamma)-s_2(\Gamma+p)\exp[(\Gamma-p)t_1]\hat{\psi}_2(\Gamma) \\ & -s_2(\Gamma-p)\exp[-(\Gamma+p)(t_2+T_2)]\hat{\psi}_2(-\Gamma)).\end{aligned} \tag{4.6}$$

Referring to equation (4.5), we note that the denominator of $\hat{\psi}_1(p)$ has two zeros. Since we know $\hat{\psi}_1(p)$ to be analytic in the finite part of $p$ plane we demand that the numerator on the right-hand side of equation (4.5) also vanish at these zeros, given by $\pm p_1 = \pm[\Gamma^2-(2\Gamma/\lambda)s_1]^{1/2}$. Exactly similar considerations apply to equation (4.6) and $\hat{\psi}_2(p)$, the zeros this time being given by $\pm p_2 = \pm[\Gamma^2-(2\Gamma/\lambda)s_2]^{1/2}$. These requirements on the right-hand sides of equations (4.5) and (4.6) yield

$$\begin{aligned}& -s_1(\Gamma+p_1)\exp[(\Gamma-p_1)t_1]\hat{\psi}_1(\Gamma)-s_1(\Gamma-p_1)\exp[-(\Gamma+p_1)(t_1+T_1)]\hat{\psi}_1(-\Gamma) \\ & \quad +\exp[(\Gamma-p_1)t_1](\Gamma+p_1)\{-1+\exp[(\Gamma-p_1)T_1]\}s_2\hat{\psi}_2(\Gamma) = 0\end{aligned} \tag{4.7a}$$

$$\begin{aligned}& s_1(\Gamma-p_2)\exp[-(\Gamma+p_2)t_2]\{1-\exp[-(\Gamma+p_2)T_2]\}\hat{\psi}_1(-\Gamma) \\ & \quad -s_2(\Gamma+p_2)\exp[(\Gamma-p_2)t_2]\hat{\psi}_2(\Gamma) \\ & \quad -s_2(\Gamma-p_2)\exp[-(\Gamma+p_2)(t_2+T_2)]\hat{\psi}_2(-\Gamma) = 0\end{aligned} \tag{4.7b}$$

$$-s_1(\Gamma-p_1)\exp[(\Gamma+p_1)t_1]\hat{\psi}_1(\Gamma)-s_1(\Gamma+p_1)\exp[-(\Gamma-p_1)(t_1+T_1)]\hat{\psi}_1(-\Gamma)$$
$$+\exp[(\Gamma+p_1)t_1](\Gamma-p_1)\{-1+\exp[(\Gamma+p_1)T_1]\}s_2\hat{\psi}_2(\Gamma)=0 \qquad (4.7c)$$

$$s_1(\Gamma+p_2)\exp[-(\Gamma-p_2)t_2]\{1-\exp[-(\Gamma-p_2)T_2]\}\hat{\psi}_1(-\Gamma)$$
$$-s_2(\Gamma-p_2)\exp[(\Gamma+p_2)t_2]\hat{\psi}_2(\Gamma)$$
$$-s_2(\Gamma+p_2)\exp[-(\Gamma-p_2)(t_2+T_2)]\hat{\psi}_2(-\Gamma)=0. \qquad (4.7d)$$

Equations (4.7*a*)–(4.7*d*) constitute a set of homogeneous equations in $\hat{\psi}_1(\Gamma)$, $\hat{\psi}_1(-\Gamma)$, $\hat{\psi}_2(\Gamma)$ and $\hat{\psi}_2(-\Gamma)$ and nontrivial solutions will result only if the determinant of the coefficients vanishes. The equation of the determinant to zero gives the eigenvalue equation. However, we now take that determinant $D(\xi)$ ($\xi = 1/\lambda$) and proceed to construct an entire function from it in order to achieve the generating function. We note that the determinant $D(\xi)$ is given by

$$D(\xi)=\begin{vmatrix} -s_1(\Gamma+p_1)\,e^{(\Gamma-p_1)t_1} & -s_1(\Gamma-p_1)\,e^{-(\Gamma+p_1)(t_1+T_1)} & e^{(\Gamma-p_1)t_1}(\Gamma+p_1)(-1+e^{(\Gamma-p_1)T_1})s_2 & 0 \\ 0 & s_1(\Gamma-p_2)\,e^{-(\Gamma+p_2)t_2}(1-e^{-(\Gamma+p_2)T_2}) & -s_2(\Gamma+p_2)\,e^{(\Gamma-p_2)t_2} & -s_2(\Gamma-p_2)\,e^{-(\Gamma+p_2)(t_2+T_2)} \\ -s_1(\Gamma-p_1)\,e^{(\Gamma+p_1)t_1} & -s_1(\Gamma+p_1)\,e^{-(\Gamma-p_1)(t_1+T_1)} & e^{(\Gamma+p_1)t_1}(\Gamma-p_1)(-1+e^{(\Gamma+p_1)T_1})s_2 & 0 \\ 0 & s_1(\Gamma+p_2)\,e^{-(\Gamma-p_2)t_2}(1-e^{-(\Gamma-p_2)T_2}) & -s_2(\Gamma-p_2)\,e^{(\Gamma+p_2)t_2} & -s_2(\Gamma+p_2)\,e^{-(\Gamma-p_2)(t_2+T_2)} \end{vmatrix} \qquad (4.8)$$

It can be observed that $D(\xi)$ is not an entire function of $\xi$ because it does not retain its original value if we go around any arbitrary closed contour containing the origin of the $\xi$ plane. This is due to the multiple valued nature of $p_1$ and $p_2$, the multiple-valued nature reflecting itself in the possible flipping of $p_1$ to $-p_1$, or $p_2$ to $-p_2$, or both. We divide $D(\xi)$ by $p_1p_2$ and obtain an entire function $P(\xi)$ of its argument:

$$P(\xi)=D(\xi)/p_1p_2. \qquad (4.9)$$

From equations (4.8) and (4.9) it is not difficult to show that $P(0)$ and $P(-\bar{I})$ are given by

$$P(0)=-16\Gamma^2s_1^2s_2^2 \qquad (4.10)$$

$$P(-\bar{I})=-16\Gamma^2s_1^2s_2^2\exp[-\Gamma(T_1+T_2)]\left[\cosh\hat{p}_1T_1+\frac{1}{2}\left(\frac{\hat{p}_1}{\Gamma}+\frac{\Gamma}{\hat{p}_1}\right)\sinh\hat{p}_1T_1\right]$$
$$\times\left[\cosh\hat{p}_2T_2+\frac{1}{2}\left(\frac{\hat{p}_2}{\Gamma}+\frac{\Gamma}{\hat{p}_2}\right)\sinh\hat{p}_2T_2\right]$$
$$+4s_1^2s_2^2(\Gamma^2-\hat{p}_1^2)(\Gamma^2-\hat{p}_2^2)\exp[\Gamma(T_1-T_2)]\exp[2\Gamma(t_1-t_2)]$$
$$\times\frac{\sinh\hat{p}_2T_2}{\hat{p}_2}\,\frac{\sinh\hat{p}_1T_1}{\hat{p}_1} \qquad (4.11)$$

where

$$\hat{p}_1 = (\Gamma^2 + 2\Gamma \bar{I} s_1)^{1/2}$$

$$\hat{p}_2 = (\Gamma^2 + 2\Gamma \bar{I} s_2)^{1/2}.$$

Whence, from equation (3.12) we get the generating function $Q(s_1, s_2)$:

$$Q(s_1, s_2) = \Bigg\{ \exp[-\Gamma(T_1 + T_2)] \left[ \cosh \hat{p}_1 T_1 + \frac{1}{2}\left(\frac{\hat{p}_1}{\Gamma} + \frac{\Gamma}{\hat{p}_1}\right) \sinh \hat{p}_1 T_1 \right] \times \left[ \cosh \hat{p}_2 T_2 + \frac{1}{2}\left(\frac{\hat{p}_2}{\Gamma} + \frac{\Gamma}{\hat{p}_2}\right) \sinh \hat{p}_2 T_2 \right] - \frac{1}{4}\left(\frac{\Gamma}{\hat{p}_1} - \frac{\hat{p}_1}{\Gamma}\right)\left(\frac{\Gamma}{\hat{p}_2} - \frac{\hat{p}_2}{\Gamma}\right) \exp[\Gamma(T_1 - T_2) + 2\Gamma(t_1 - t_2)] \sinh \hat{p}_1 T_1 \sinh \hat{p}_2 T_2 \Bigg\}^{-1}. \tag{4.12}$$

This completes the derivation of the generating function for the two-time case for gaussian–lorentzian light. The probability distribution $p(n_1, t_1, t_1 + T_1; n_2, t_2, t_2 + T_2)$ and the factorial moments $\langle n_1^{l_1} n_2^{l_2} \rangle$ are given by

$$p(n_1, t_1, t_1 + T_1; n_2, t_2, t_2 + T_2) = \frac{(-\alpha)^{n_1}}{n_1!} \frac{(-\alpha)^{n_2}}{n_2!} \left( \frac{\partial^{n_1}}{\partial s_1^{n_1}} \frac{\partial^{n_2}}{\partial s_2^{n_2}} \right) Q(s_1, s_2) \Big|_{\substack{s_1 = \alpha \\ s_2 = \alpha}} \tag{4.13}$$

$$\langle n_1^{l_1} n_2^{l_2} \rangle = (-\alpha)^{l_1} (-\alpha)^{l_2} \left( \frac{\partial^{l_1}}{\partial s_1^{l_1}} \frac{\partial^{l_2}}{\partial s_2^{l_2}} \right) Q(s_1, s_2) \Big|_{\substack{s_1 = 0 \\ s_2 = 0}} \tag{4.14}$$

$\alpha$ being the quantum sensitivity of the detector. We note that just as the single-time generating function can be experimentally measured at the point 1 (see Kelly and Blake 1971), the generating function given by (4.12) can also be measured experimentally at $s_1 = 1$, $s_2 = 1$; this value is nothing but the probability that zero counts are registered during both the sample times $(t_1, t_1 + T_1)$ and $(t_2, t_2 + T_2)$.

When the sample times $T_1$ and $T_2$ are both equal to $T$ (say), we can dispense with the condition that they should fall within $(0, T)$ (as indeed we must!). This condition previously imposed is essentially a device to preserve the normalization of the eigenfunctions $\phi$ (or equivalently to see that the mean intensity of the beam is the same during different sample times). When $T_1 = T_2 = T$ the same normalization still holds but $t_1$ and $t_2$ can occur at any times whatsoever. If we now take the two sample times as centred around $\tau$ and $0$ $(\tau < 0)$ we recover the two-time generating function first derived by Jakeman (1970). This is obtained from equation (4.12) by putting $t_1 = \tau - \frac{1}{2}T$, $t_2 = -\frac{1}{2}T$ and $T_1 = T_2 = T$. The result is

$$Q(s_1, s_2) = \Bigg\{ e^{-2\Gamma T} \left[ \cosh \hat{p}_1 T + \frac{1}{2}\left(\frac{\Gamma}{\hat{p}_1} + \frac{\hat{p}_1}{\Gamma}\right) \sinh \hat{p}_1 T \right] \left[ \cosh \hat{p}_2 T + \frac{1}{2}\left(\frac{\Gamma}{\hat{p}_2} + \frac{\hat{p}_2}{\Gamma}\right) \sinh \hat{p}_2 T \right] - \frac{1}{4} e^{2\Gamma\tau} \left(\frac{\Gamma}{\hat{p}_1} - \frac{\hat{p}_1}{\Gamma}\right)\left(\frac{\Gamma}{\hat{p}_2} - \frac{\hat{p}_2}{\Gamma}\right) \sinh \hat{p}_1 T \sinh \hat{p}_2 T \Bigg\}^{-1}. \tag{4.15}$$

When the intervals overlap, one can still retain the forms of equations (2.1) and (2.11) by defining new $s$'s. All other calculations carry through.

The method outlined here can also be applied to a mixture of an incoherent gaussian beam and a coherent single-mode beam with minor modifications. For beams with nonlorentzian profiles, the techniques outlined here combined with the method used by Srinivasan and Sukavanam (1971) should yield the desired explicit expression for the generating function.

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